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Large deformation and failure simulations for geo-disasters using smoothed particle hydrodynamics method

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ABSTRACT

Geo-disasters result in serious loss of life and property, and prediction and prevention of these disasters is of great importance. The smoothed particle hydrodynamics (SPH) method, a mesh-less hydrodynamics technique, was applied to the modeling of large deformation and post-failure behavior of geomaterials in geo-disasters with some success. The main aim of this paper is to provide a general view of SPH applications for solving a range of large deformation and failure problems, such as dam breaks, slope failure, soil liquefaction, seepage damage, dynamic erosion, underground explosions and rock breakage. Rather than attempting to cover every application found in the technical literature, this review selects some typical examples and describes them in detail.

Contents

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1. Introduction

Geo-disasters accompanied by large deformation and failure of geomaterials are a regular occurrence around the world. These include landslides, debris flow, dam breaks, soil liquefaction, seepage damage, and dynamic erosion. Such disasters cause serious damage to infrastructure, resulting in casualties and high economic losses. To reduce the damage, one of the priorities for governments and researchers is to determine the probability of geo-disasters occurring, devise hazard maps, and take protective measures.

Numerical simulation is a powerful tool, playing an increasingly important role in solving complex problems. Grid- or mesh-based numerical methods, such as the finite difference method (FDM) and the finite element method (FEM) have been widely applied to various areas of geomechanics. For example, Crosta et al. (2003, 2004, 2006) used the FEM model to simulate flow-like landslides; Jie et al. (2004) presented an FDM model to analyze steady seepage. All these methods were effective in solving partial differential equations (PDEs), and obtained many interesting results in studies of geo-disaster cases. Despite the success of grid methods, the use of mesh may sometimes lead to numerical difficulties (e.g., severe mesh winding, twisting, and distortion) in predicting geo-disasters accompanied by extremely large deformations, free surfaces, deformable boundaries, moving interfaces, and crack propagation.

To overcome these numerical difficulties, mesh-free methods have been developed. Smoothed particle hydrodynamics (SPH) is a recently-developed mesh-free method based on a pure Lagrangian description. As a mesh-free technique, the main advantage of SPH is that it bypasses the need for a numerical grid, therefore avoiding severe mesh distortions caused by large deformation. Hence, the SPH method has been successfully applied in a range of fields including astrophysics (Monaghan and Lattanzio, 1985; Monaghan, 1992; Curir and Mazzel, 1999; Laibe et al., 2008; Hubber et al., 2011; Valdarnini, 2011), hydrodynamics (Cleary et al., 2006a; Fang et al., 2006; Oger et al., 2006; Tartakovsky and Meakin, 2006; Xiong et al., 2006), hypervelocity impacts and collisions (Johnson et al., 1993; Michel et al., 2006; Sekine et al., 2007; Guida et al., 2011; Marrone et al., 2011), metal manufacturing (Bonet and Kulasegaram, 2000; Cleary et al., 2006b; Prakash et al., 2009), flow slides in landfills (Huang et al., 2013), and mining engineering (Cleary et al., 2006c; Fernandez et al., 2011). More recently, a variety of corrected and improved SPH methods have been developed, including: discontinuous smoothed particle hydrodynamics (DSPH) (Xu et al., 2013), corrected smooth particle hydrodynamics (CSPH) (Rodriguez-Paz and Bonet, 2005), regularized smoothed particle hydrodynamics (RSPH) (Borve et al., 2005), and adaptive smoothed particle hydrodynamics (ASPH) (Attwood et al., 2007; Sigalotti et al., 2009). These methods all enhance the consistency and stability of the SPH method and are widely applied in various research and analysis programs.

In view of the powerful capabilities of the SPH method in large deformation analysis, this method has recently been introduced to geo-disaster prediction and simulation. The objective of this paper is to detail some features associated with SPH simulations for large deformation and post-failure behavior of geomaterials. We first present a brief overview of the basic principles of the SPH method, followed by an overview of SPH applications to geological disaster cases and its related literature, and then conclude by discussing the achievements of the SPH application in geo-disaster prediction and the method's future direction.

2. Overview of the SPH method

2.1. SPH approximation techniques

The SPH method, first developed for astrophysical applications, is a novel mesh-free particle method based on a pure Lagrangian description (Gingold and Monaghan, 1977; Lucy, 1977). The basic idea behind this method is to provide stable and accurate numerical solutions for PDEs or integral equations using a group of arbitrarily distributed particles carrying field variables, such as mass, density, energy and stress tensors.

The SPH method is based on interpolation theory. The governing equations, in the form of PDEs with field variables, can be transformed into SPH form through two main steps. The first step is to produce continuous forms of functions using integral representations. This is accomplished by applying interpolation functions. This step is usually called the kernel approximation, and the interpolation function is called the smoothing function or smoothing kernel function. For example, a function $f(x)$ at location x could be rewritten in a continuous form:

$$\langle f(x)\rangle = \int_{\Omega} f\left(x^{'}\right) W\left(x-x^{'}, h\right) dx^{'}, \tag{1}$$

where the angle brackets < > denote a kernel approximation, x represents the location vector of the particle, Ω is the volume of the integral that contains x, and x' is a neighboring particle in the support area. The parameter h defines the size of the kernel support, known as the smoothing length. W denotes the smoothing function.

The second step is to represent the problem domain using a set of discrete particles within the influence area of the particle at x, and then to estimate the field variables for those particles. This step is named the particle approximation, and is expressed as follows:

$$\langle f(x)\rangle = \sum_{j=1}^{N} m_j \frac{f\left(x_j\right)}{\rho_j} W\left(x-x_j, h\right), \tag{2}$$

where N is the total number of neighboring particles, m is the mass, and ρ is the density.

The particle approximation states that the value of a function at a particle can be estimated by the average value of all the particles in the support domain. This step makes the SPH method simple without requiring a background mesh for numerical integration.

2.2. Special topics

2.2.1. Solid boundary treatment

In most problems of geological engineering, the domain of interest is bounded. The bounding domain, which is usually stationary, might be a rigid body enclosing the fluid or solid matter. A number of techniques have been proposed to treat the solid boundary condition. Free-slip boundaries were used in SPH simulations of free surface flows, with boundary particles that exert strong repulsive forces to prevent SPH particles from penetrating the solid surface (Monaghan, 1994). Those boundary particles do not contribute to the density of the free SPH particles. Libersky et al. (1993) introduced ghost particles with opposite velocity to reflect a symmetrical surface boundary condition and later proposed a more general treatment (Randles and Libersky, 1996); all the ghost particles were assigned the same boundary field variable to calculate the values of the interior particles. Morris et al. (1997) proposed the non-slip boundary condition. In this technique, a tangent plane to the boundary surface is defined, and the velocity on the plane itself is assumed to be zero. Extrapolating the velocity of the fluid particles across the tangent plane, the velocity of each boundary particle would be $V_b = -(d_b/d_f)V_f$, where d_b and d_f are the shortest distances from the boundary particle and the fluid particle to the tangent plane, respectively. The difference between the fluid and boundary particle velocities is then $V_{bf} = (1 + d_b/d_f)V_f$, which can be used to calculate the viscous force. On this basis, a new boundary treatment method was presented, called the multiple boundary tangent (MBT) method for more complex boundary geometries (Yildiz et al., 2009). Fluid particles in the influence domain of the boundary particle were mirrored with respect to the tangent line of the corresponding boundary particle.

During the SPH summation, the mirrored particle was given the same mass, density, and transport parameters as the fluid particle, and its field values obtained according to the type of boundary condition implemented. This boundary treatment was verified by some benchmark problems.

2.2.2. Material interface treatment

Geo-disasters involve intricate problems concerning complex material interfaces between different geological materials. The capability to track material interfaces is among the advantages of mesh-free methods. According to the conventional SPH algorithm, particles of one material can influence particles of another material and be influenced by them. This interaction between particles from different materials can introduce shear and tensile stress, which prevents sliding and separation of dissimilar materials. The main issue of the interface treatment is the density discontinuity when crossing the interface and special algorithms are required to simulate this. The major steps of the interface treatment in the SPH simulation include (1) interface identification, (2) contact detection, and (3) repulsive contact force application. Campbell et al. (2000) presented a particle-to-particle contact algorithm for the SPH method, using a penalty formulation to enforce the contact condition. They compared several equations for the penalty force calculation and the best approach was determined. Liu et al. (2003a) applied a penalty force as a Lennard–Jones potential on the two approaching particles along the line joining the centers of the two particles, combined with a highly peaked repulsive viscous force. This method was successfully applied to simulate an underwater explosion. Mutsuda et al. (2008) solved the fluid structure interaction problem by automatically capturing the solid interface by overlapping particles on fixed grids. Grenier et al. (2009) presented a Hamiltonian interface SPH formulation based on the Lagrangian variational approach introduced by Bonet and Lok (1999); the normalized Shepard kernel was used to ensure the preservation of the discontinuity of the density across the interface. A number of test cases of multi-fluid and free surface flows were simulated to validate the formulation. Recently, a new fluid–solid interface algorithm was presented (Eghtesad et al., 2012). The continuity equation for both fluid and solid particles is improved by a renormalization scheme—the corrected smooth particle method (CSPM)—and the coupled field equations are solved as the momentum equations for the particles near the interface. The new interface scheme prevents the penetration of fluid and solid particles and significantly improves the gap between the fluid and solid boundaries in the contact region.

2.3. Numerical implementation

The numerical implementation of the SPH method is easier than that of traditional grid-based numerical methods. In our earlier paper (Huang et al., 2011a) we explained the typical serial SPH program structure through a processing flowchart combined with basic formulas. Moreover, Liu and Liu (2003) presented a 3D SPH code with detailed comments. Both methods are helpful to understand the SPH algorithm, and show promise in application of practical engineering problems.

2.4. Advantages

The advantages of the SPH method over traditional grid-based numerical methods are:

1. SPH is a numerical method based on a pure Lagrangian description. The motion of the particles can be traced and the features of the entire physical system can be easily obtained. Therefore, it is easier to identify the free surfaces, moving interfaces, and deformable boundaries using the SPH method than by Eulerian methods. The time history of the field variables at each material point can also be obtained in the simulation. Therefore, SPH is an ideal choice for modeling free surface and interfacial flow problems.
2. As a Lagrangian method, the SPH code is conceptually simpler than grid-based methods and should be faster as no convective term exists in the related partial differential equations.
3. In the SPH method, the object under consideration can be discretized into a series of particles without using a grid/mesh. Compared with grid-based methods, this distinct mesh-free feature can therefore process larger local distortion because the connectivity between the particles is generated as part of the computation and can change with time. This feature has been utilized in many applications in solid mechanics such as underground explosions, metal forming, high velocity impact, crack growth, and fragmentation.
4. As a particle method, discretization of complex geometry is relatively simpler as only an initial discretization is required. The refinement of particles would be much easier to perform than the mesh refinement.
5. SPH guarantees conservation of mass without extra computation since the particles themselves represent mass. Pressure is computed from the weighted contributions of the neighboring particles rather than by solving the linear systems of equations.
6. SPH is suitable for problems where the material is not a continuum. Therefore, it is a valuable tool for numerical simulation of problems in bio- and nano-engineering at micro and nano scale.

2.5. Defects and modifications

While the favorable features of the SPH method and its applications to geotechnical engineering have been noted, drawbacks, such as inconsistency, tensile instability, and zero energy modes, have also been identified.

2.5.1. Inconsistency

The SPH method in its continuous form is inconsistent close to the boundaries because of the incompleteness of the kernel support. Morris (1996) and Belytschko et al. (1996) identified a particle inconsistency problem that can lead to low accuracy in the SPH solution. Various solutions have been proposed to restore the consistency and improve the accuracy of the SPH method. Randles and Libersky (1996, 2000) use the inconsistency in approximating the smoothing function and its derivatives to offset the inconsistency in approximating the field function and its derivatives. Vignjevic et al. (2000) implemented kernel normalization and correction in the corrected normalized smooth particle hHydrodynamics (CNSPH) method which is first-order consistent. These proposed modifications are based on either the kernel approximation or on the particle approximation. Recently, Chen and Beraun (2000) presented a corrective smoothed particle method (CSPM) based on the Taylor series expansion of the SPH approximation of a function. Liu et al. (2003b) improved the CSPM in the discontinuous SPH (DSPH) methods to resolve problems with discontinuity such as in shock waves. Other notable modifications or corrections of the SPH method that ensure first-order consistency include the element free Galerkin method (EFGM) (Belytschko et al., 1994), the reproducing kernel particle method (RKPM) (Liu et al., 1995), the moving least square particle hydrodynamics (MLSPH) method (Dilts, 1999), and the meshless local Petrov Galerkin (MLPG) method (Atluri and Zhu, 2000). These methods allow the restoration of consistency of any order by means of a correction function.

2.5.2. Tensile instability

Tensile instability is a numerical problem that appears in the conventional SPH method and greatly limits its application in Geological Engineering. This instability manifests itself as a clustering of the particles when they are under tensile stress. Swegle et al. (1995) first revealed this phenomenon and provided a stability criterion, noting that the tensile instability is closely related to the second derivative of the smoothing kernel function. Various remedies were proposed for this problem. Dyka and Ingel (1995), Dyka et al. (1997) added a series of additional stress points in the support domain other than the normal particles in a one-dimensional algorithm to remove the tensile instability. Randles and Libersky (2000, 2005) used these stress points to stagger all the SPH particles and extended this approach to multi-dimensional space. Monaghan (2000) removed the tensile instability by introducing an artificial stress. This method has been widely applied in Geological Engineering problems (Bui et al., 2008; Das and Cleary, 2010; Karekal et al., 2011). Bonet and Kulasegaram (2001) discussed the problem of tension instability of the SPH method, and stated that tension instability is a property of a continuum where the stress tensor is isotropic and the pressure is a function of the density. They demonstrated that a stable solution can be obtained using Lagrangian CSPH without the need for any artificial viscosity. More recently, a new method to avoid the tensile instability was presented with two sets of master and slave nodes being used (Blanc and Pastor, 2012a, 2013).

2.5.3. Zero-energy modes

Zero-energy modes, first identified in an SPH solution by Swegle et al. (1994), represent modes of deformation characterized by a pattern of nodal displacement that produces zero strain energy. This problem can produce spurious oscillations and lead to a degradation of the solution. The main cause for this condition is that all the field variables and their derivatives are calculated at the same locations, so that an alternating field variable has a zero gradient at the particles. Two types of solutions can be found in the existing literature: dissipation of spurious modes, and an alternative discretization that does not evaluate the variables and their derivatives at the same points. For example, an artificial stress was used to preclude instability (Randles and Libersky, 2000). Two different sets of particles were used to evaluate the stresses and velocities at separate points (Vignjevic et al., 2000). In addition, a stabilized updated Lagrangian formulation was incorporated in the SPH model to overcome the problem of zero-energy modes (Vidal et al., 2007).

Besides the main shortcomings discussed above, the SPH method has some other defects. For example, the conventional SPH method can only be used to simulate compressible fluid, a problem that was solved by the weakly compressible SPH (WCSPH) and the incompressible-SPH (ISPH) methods, which will be discussed in detail in Section 3.2. Another well-known problem is that the amount of smoothing needed for stability may dampen the short-wavelength structure (Hicks and Liebrock, 1999) and smooth out strong shocks. This problem could be detrimental in the analysis of certain geophysical processes that involve shock waves. Reformulation of standard SPH arithmetic for strong shock simulation has been proposed by Monaghan (1997), Inutsuka (2002), and Cha and Whitworth (2003). The common feature of these methods is the combination of standard SPH with Riemann solvers. Recently, Sigalotti et al. (2009) presented an adaptive SPH (ASPH) method for strong shocks. This method relied on an adaptive density kernel estimation (ADKE) algorithm, which allows the smoothing length to vary locally in space and time so that the minimum necessary smoothing is applied in regions of low density.

3. SPH applications in geo-disasters

Over the last few years, the SPH method has been extended to a wide range of problems in both fluid and solid mechanics because of its strong ability to incorporate complex physical concepts into SPH formulations. A variety of SPH models have been proposed and applied to specific topics in geo-disasters, including dam breaks and coastal engineering, flow-like landslides, the lateral spread of liquefied soil, seepage failure, dynamic erosion, underground explosions, and rock breakage. The feasibility and reliability of such models was verified successfully through comparisons with laboratory experiments, analytical solutions, and simulations with other methods. In the following section, some recent applications of SPH methods in geo-disasters are described.

3.1. Dam breaks and coastal engineering

Because of the successful applications of the SPH method in hydrodynamics, most SPH simulation studies of geo-disaster topics are concentrated in fields related to fluid dynamics, such as dam breaks and coastal engineering.

During a dam failure, a huge amount of water stored in the reservoir suddenly rushes downstream, destroying trees, dikes, buildings, and bridges. It is important to predict the effects of catastrophic dam-break floods to minimize the human and financial toll. Because of the mesh-free, Lagrangian and particle nature of the SPH method, the technique has been applied widely to studies related to dam breakage. Many inter-related aspects of dam-break problems have been investigated, including approaches to the treatment of free surface flow (Chang et al., 2011; Koh et al., 2012), three-dimensional (3D) assessments of dam-break disasters (Roubtsova and Kahawita, 2006; Ferrari et al., 2010), the treatment of boundary conditions (Ata and Soulaimani, 2005; Crespo et al., 2007), the differences between Newtonian and non-Newtonian flow features (Shao and Lo, 2003), the analysis of interfaces in multi-phase flow (Colagrossi and Landrini, 2003), and multiphase models for highly erosive flow (Shakibaeinia and Jin, 2011). In these simulations, the unique advantages of the SPH method for dealing with the free surface and moving boundary problems were fully embodied. A simple example of SPH application to dam-break analysis was given by Wang and Shen (1999) who simulated one-dimensional inviscid dam-break flows and conducted depth-averaged analysis. The unusual feature of their model is that the length of the discrete parcels varies with changing flow conditions. Comparisons with analytical and other numerical solutions showed that the method was robust and especially suited to solving problems with sharp moving fronts. Fig. 1 shows an SPH simulation of free-surface flow generated by a dam break presented in our previous work (Huang et al., 2012). The time history of the position of the simulated surge front agreed precisely with the experimental results provided by Martin and Moyce (1952) (see Figure 2), which demonstrates the application of the SPH method to dam-break flows.

Large sea waves occasionally break through coastal defenses and travel inland for long distances, resulting in damage to infrastructure and loss of life. Therefore, an important aspect of any mitigation effort is to predict the process of wave generation, shoreward propagation, arrival at the shoreline, building up of water height and the nature of wave breaking. The successful application of SPH methods to dam-breaking has provided a foundation for the solution of fluid–structure interaction problems found in coastal engineering.

Gomez-Gesteira et al. (2005) established a numerical model within the framework of an SPH method to analyze the phenomenon of wave overtopping on ships and offshore platforms. Some complex phenomena were successfully reproduced, including 1) the initial continuous flow, 2) flow separation when hitting the deck, 3) different wave behavior above and below the deck, 4) the formation of a jet after overtopping, and 5) further flow restoration. All of the simulations accurately matched experimental observations. Mutsuda et al. (2008) applied an SPH method to study the interaction between waves and engineered coastal structures. The solid interface was automatically captured by particles overlapping on fixed grids. In their model, the deformation, movement, and failure process of elastic and rigid coastal structures subjected to impact from sea waves was calculated with smoothness, efficiency, and accuracy. Lo and Shao (2002) developed an SPH model coupled with a large eddy simulation (LES) approach to analyze the mechanics of near-shore solitary waves and to address the typical problem of a solitary wave rising and falling against a vertical wall. Following this, Shao et al. (2006) presented a similar SPH-LES model combined with a sub-particle scale (SPS) turbulence model for the treatment of turbulence associated with wave breaking. The

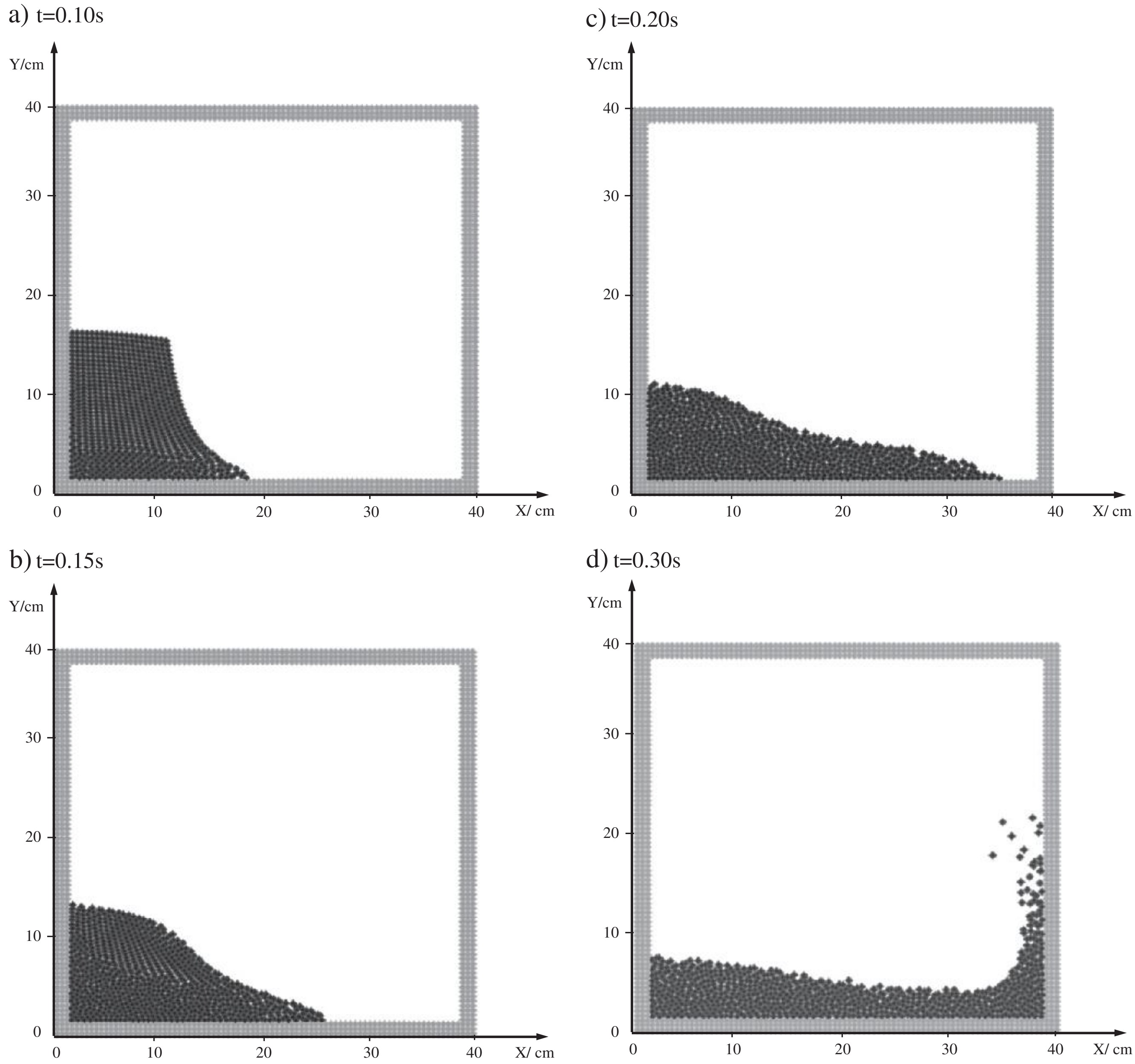

Fig. 1. Free-surface flow generated by the break of a dam (based on Huang et al., 2012).

configurations and overtopping characteristics of different types of waves (e.g., velocity fields and turbulent eddy viscosity distributions) were predicted. More recently, Shao (2010) added more friction forces into the Navier–Stokes equations within an SPH framework to examine flow friction in porous media and consider the interface between waves and a breakwater covered with a layer of geomaterial. This model has been validated for solitary and periodic waves damped over a porous bed and applied to the case of a breaking wave running up and over a breakwater protected by a porous layer of geomaterial.

These proposed models address some significant problems in coastal engineering, such as free surface, moving boundaries, and solid–liquid coupling. They accurately reproduce the phenomena of dam breaking, wave dynamics (wave generation, breaking, and interaction with structures), and the failure of breakwaters and their foundations. The results provide significant foundation for the design of offshore structures and the assessment of dam-break or tsunami disasters.

3.2. Slope failure and landslides

Natural slopes in soil and soft rock have become more vulnerable as human activities gradually extend into mountainous regions. Understanding the failure mechanism and post-failure behavior of slopes is important to determine potential risk areas and devise hazard maps. The SPH method is an important tool for modeling slope failure. For example, Bui et al. (2008) presented an SPH framework for the stability analysis of a slope with reinforcing piles. They proposed an algorithm to deal with the problem of soil–structure interaction, which adopts a coupling condition on the interface between soil and structure associated with a penalty force applied to different material particles near the interface. This model can simulate the following phenomena and analyze the mechanism in the following four cases: 1) the development of shear bands by investigating the accumulated plastic strain contour plot, 2) the gross discontinuity of soil after failure, 3) stress distribution

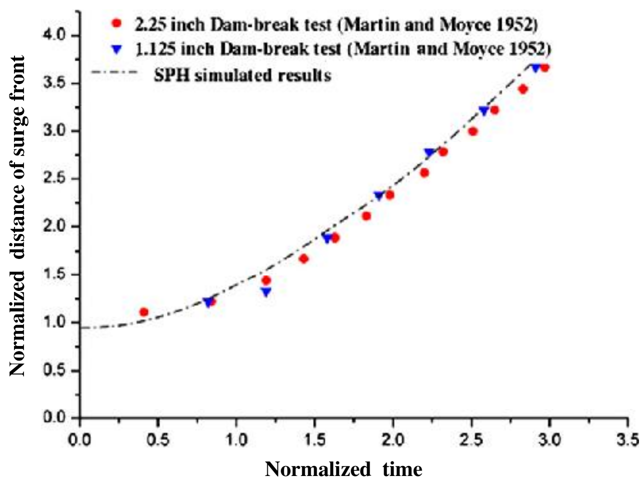

Fig. 2. Comparison of SPH simulation and experimental results for dam break data (based on Huang et al., 2012).

on the reinforcing pile, and 4) the bending mechanism of the reinforcing pile. This is the first study known to implement an elastoplastic soil constitutive model (the Drucker–Prager model with a non-associated flow rule) into an SPH model to describe plastic soil behavior. Another highlight of this research is proposing a well-performing contact algorithm to deal with soil–structure interactions. However, particle deficiency near the solid boundary may be one of the toughest problems of the SPH method when applied to simulate elastoplastic material. Although some solutions have been proposed, the low precision caused by the lack of particle coverage near the solid boundary in this method needs to be addressed further.

Flow-like landslides often result in catastrophic events because of their relatively long run-out distances and high velocities. A prediction of the run-out, velocity, and impact force of landslides is important so that adequate protective measures can be taken. A numerical model for the dynamic analysis of rapid landslide motion across 3D terrain has been developed (McDougall and Hungr, 2004). The depth-integrated equations were used to govern the mass and momentum balance of a column of earth material moving with the landslide; these governing equations were discretized by the SPH method. The model was tested using an analytical solution to the classical dam-break problem, and conducting a series of laboratory experiments; it was then applied in the analysis of the Frank Slide, Canada, with promising results. The model has many unique features, such as the ability to account for nonhydrostatic and anisotropic internal stress states, material entrainment along the slide path, and rheology variation. Its path material entrainment algorithm was described in detail in McDougall and Hungr (2005), and the importance of this capability was demonstrated using a back-analysis of the 1999 Nomash River landslide, Canada. Since then, depth-integrated models have been frequently used to model flow-like landslides; of particular note is the pioneering work of Pastor et al. (2009). They proposed a depth-integrated model in combination with the SPH method to simulate the propagation of flow-like catastrophic landslides. The Biot–Zienkiewicz model in the velocity–pressure version was introduced to consider the coupling effect between the solid and fluid phases. As an example, the propagation stage of the catastrophic landslide of May 1998 in the Tuostolo Basin, located in the Campania region of southern Italy, was simulated, and the modeled results were found to coincide with available field data. Later on, this model was applied to simulate the propagation stage of a lahar that occurred in 2001 at the Popocatépetl volcano, Mexico (Haddad et al., 2010); the trajectory, velocities, depths and run-out distances of the fluidized materials were correctly predicted. The sensitivity of the proposed model to the rheological parameters was studied. The results showed that the viscosity had a strong influence on the flow velocity while the yield strength affected mainly the run-out distance. Our group presented an SPH model to predict the run-out of flow-like landslides from the viewpoint of hydrodynamics (Huang et al., 2012); visual software with user-friendly interfaces was successfully developed, improving the simulation efficiency (Huang et al., 2011a). The propagation of typical flow-like landslides induced by the 12 May 2008 Wenchuan earthquake, China, was modeled and the simulated run-outs had a high degree of similarity with surveyed landslide configurations (Figure 3). However, in our model, the landslide material was assumed as a Bingham fluid, and the interaction of the soil and water was ignored, which may have introduced a certain error. Another limitation of our model is that it simulated landslide motion along a user-prescribed two-dimensional (2D) path. However, landslides travel across three-dimensional (3D) terrain, and may change direction, spread or contract, and split or join in response to local topography. For example, field investigations indicated that the dynamic behavior of the Donghekou landslide, triggered by the 2008 Wenchuan earthquake, is controlled by geologic and tectonic conditions and the local geomorphological aspects of the terrain (Sun et al., 2011). Therefore, 3D modeling is needed to truly reproduce the dynamic process of flow-like landslides from slide initiation to the final cessation of slide movement. On other hand, SPH is a time-consuming numerical method. A large number of SPH particles have be used to simulate the large volumes of failure soil materials and the complex 3D terrain. This reduces the calculation efficiency; therefore, our future research will focus on improving the calculation efficiency of SPH methods and investigating parallel computing techniques.

In risk analysis, rock avalanches pose serious hazards to the growing population in mountain areas. Sosio et al. (2012) investigated the mobility of rock and debris avalanches evolving in a glacial environment using the SPH model proposed by McDougall (2006). The propagation outline, flow velocity, erosion depth, and deposit thickness were simulated. A quantitative comparison between the simulation results and field data was conducted and a good agreement was obtained. Moreover, the values of the calibrated parameters were provided through back analyses.

Landslides, either submarine or aerial, can generate surface water waves that may cause damage and loss of life in coastal areas. Predicting the extent of these waves is important for assessing the level of flooding. Currently, it is still difficult to simulate surface waves generated by a landslide because of the complex motion of an underwater slope and its interaction with water. Because of its advantages in simulating free surfaces, moving boundaries, and large deformations, the SPH method is widely used to deal with the problem of landslide impulsive waves. Schwaiger and Higman (2007) presented an SPH model to simulate the 1958 Lituya Bay rockslide, Alaska, and the resulting tsunami. The rock and water were treated in the model as a viscous and inviscid fluid, respectively, while the effect of the air was neglected. Qiu (2008) used a 2D SPH model for inviscid fluid to simulate landslide-induced waves, and the propagation of the water was accurately predicted. Numerical simulations of Tsunami wave generation were carried out by Das et al. (2009); the complex flow patterns predicted in terms of the free-surface profiles, shoreline evolution, and velocity field were in good agreement with experimental data. Capone et al. (2010) presented a rheological SPH model to investigate the solid–liquid interaction and reproduce the generation and propagation of a tsunami triggered by underwater landslides.

Landslide-generated waves involve processes of slide motion, interaction with water and air, induced water surface deformation, and air entrapment. Conventionally, the Eulerian formulation is widely used to simulate wave propagation, track the free surfaces, and capture moving interfaces, because it allows relatively easy implementation of the conservation laws of motion (Das et al., 2009). Therefore, some Eulerian techniques have been incorporated into SPH models, such as the volume of fluid (VOF) method and the level-set (LS) approach. The

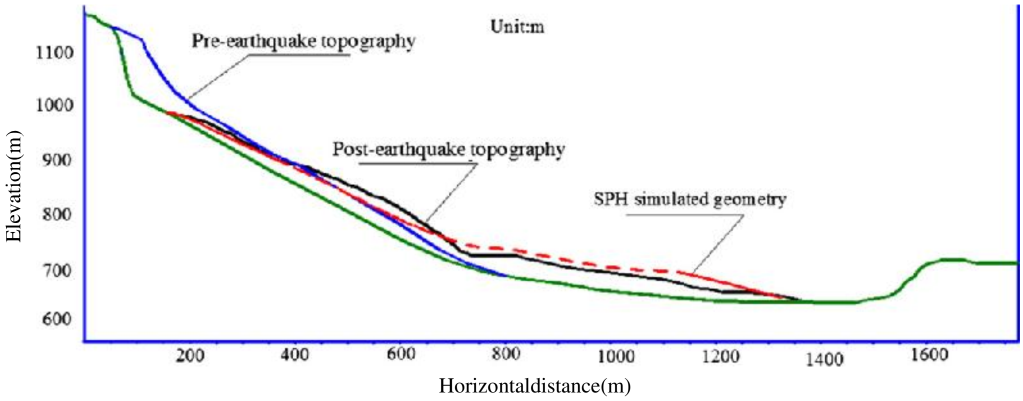

Fig. 3. Pre- and post-earthquake topographic profiles. A comparison of SPH simulation and surveyed data for the Donghekou landslide (based on Huang et al., 2012).

VOF method, which was initially introduced by Hirt and Nichols (1981), is an advection scheme—a numerical recipe that allows the programmer to track the shape and position of the interface. The most important feature of this method is the introduction of the phase function $F(x, y, t)$ for the description of the interface. Abadie et al. (2010) incorporated the VOF method into a 3D multiple-fluid model based on the SPH method for the simulation of landslide-generated waves. The wave, air, and slope were treated as different Newtonian fluids and the interactions were tracked. This model was validated by comparison with semi-empirical laws of motion and experimental data, and successfully applied in simulations of retrogressive slope failure, deformable slides, and granular flows. Another commonly used interface-capturing method is the LS approach (first proposed by Enright et al., 2002), in which the curvature and the surface tension are calculated directly. A level-set function was defined in the SPH solver (Marrone et al., 2010). This function distinguishes between nodes inside and outside the fluid domain and allows an in-depth analysis of the flow features through adequate post-processing. A two-way coupled numerical model was proposed by Losasso et al. (2008), in which dense liquid volumes and diffuse regions were simulated using the LS and SPH methods, respectively. A large ocean scene was simulated as the application of this model.

To enhance the stability and accuracy of the conventional SPH method for these types of complex free surface problems, the conventional SPH method, which was used to simulate an arbitrarily-moving compressible fluid, was extended to incompressible or nearly incompressible flow by using two different approaches. The first approach is the weakly compressible smoothed particle hydrodynamics (WCSPH) method. In this method, fluids are regarded as compressible with a sound speed that is much higher than the bulk flow speed. A stiff equation of state is used to calculate the pressure of the particles (Monaghan, 1994). This method was corrected and applied to simulate an impulsive wave generated by an underwater landslide (Ataie-Ashtiani and Mansour-Rezaei, 2009). This method is easy to program because the pressure is obtained directly from an algebraic thermodynamic equation (Monaghan, 1994). However, some drawbacks appear. First, this artificial compressibility can cause problems with sound wave reflection at the boundary area (Shao and Lo, 2003). Second, the time step is limited because the sound speed is much higher than the maximum velocity (Lee et al., 2008). These problems can be overcome by using the second approach, the incompressible-SPH (I-SPH). This method solves governing equations using prediction–correction fractional steps, and pressure is no longer a dependent variable, but can be computed from a pressure Poisson equation which satisfies the incompressibility condition. The main advantages of I-SPH lie in its easy and efficient tracking of the free surface and the ease with which it treats wall boundaries. For example, Shao and Lo (2003) presented an incompressible SPH method which was tested on a dam-break problem for Newtonian and non-Newtonian flows, and the results were in good agreement with experimental data. An I-SPH model was presented and tested using solitary waves generated by a heavy box falling into water (Ataie-Ashtiani and Shobeyri, 2008). Applying this model, a submerged rigid wedge sliding along an inclined surface was simulated. The computational I-SPH results were in good agreement with the experimental data. In addition, the proposed modeling method was used to simulate the flow of gravel mass sliding along an inclined plane, accurately capturing the wave profiles. Recently, a similar fractional step technique, first proposed by Chorin (1968), was incorporated into the SPH model to deal with coupled problems in geomechanics (Blanc and Pastor, 2011, 2012b). Comparisons of the I-SPH algorithm with the classical WCSPH method were presented by Lee et al. (2008), who showed that I-SPH yields much more reliable results than WCSPH. More recently, Shadloo et al. (2012) presented a comparative study of the WCSPH and I-SPH methods by providing numerical solutions for fluid flow over a square obstacle; they indicated that the WCSPH method produced numerical results as accurate and reliable as those of the I-SPH method. Szewc et al. (2012) presented a thorough comparison of these two incompressibility treatments. Their results showed that the I-SPH method suffered from density accumulation errors; a correction algorithm was used to improve the accuracy. To the best of our knowledge, no direct comparison between the conventional SPH and these two incompressibility treatments has been performed; hence, the advantages of the I-SPH and WCSPH method over the conventional SPH method cannot be presented.

Unlike traditional numerical methods based on solid mechanics, the SPH models mentioned above analyze the large deformation and post-failure behavior of slopes from a fluid mechanics point of view, and provide a completely new and effective approach for run-out prediction in addition to the empirical methods. Unlike applications in dam break studies, complex constitutive models of geomaterials have been imported into the SPH framework. For example, highly deformed soil material was modeled as a viscoplastic fluid named Bingham fluid (Ataie-Ashtiani and Shobeyri, 2008; Pastor et al., 2009; Capone et al., 2010; Haddad et al., 2010; Huang et al., 2012) and the model's sensitivity to the rheological parameters was evaluated (Haddad et al., 2010). The Drucker–Prager model with a non-associated plastic flow rule was introduced to describe the elasto-plastic soil behavior, and the solid structure was simulated as an elastic-perfectly plastic material using the Von-Mises yield criterion (Bui et al., 2008). A simpler semi-empirical approach based on the concept of "equivalent fluid" was used in a new SPH model by McDougall and Hungr (2004) in a landslide study. The landslide material was governed by a simple rheology, and its parameters were selected based on the back-analysis of full-scale landslides. The incorporation of these complex constitutive models in the SPH framework promotes the application of this method in geo-disasters. Moreover, in the simulation of submarine landslides, the interactions of different fluids were taken into account and tracked.

3.3. Liquefaction

Liquefaction is a phenomenon that occurs mainly in loose saturated sands as a result of earthquakes. Subsoil lateral spreading due to liquefaction can result in major damage to buildings and infrastructure. In numerical analyses, grid-based numerical methods such as FEM sometimes face a mesh distortion problem because of the difficulty in handling extremely large deformation. To overcome this limitation, Naili et al. (2005a) introduced a 2D SPH-based numerical model in a fluid dynamic framework to analyze the lateral spreading induced by liquefaction. The liquefied subsoil was considered to be a non-Newtonian fluid by means of a Bingham fluid model. Under this hypothesis, the soil is capable of resisting any shear below a yield defined by the residual shear strength. The ability of the method to reproduce the free surface shape and obtain a time history of flow velocities was validated through comparisons with “shake table” experiment results. Through the application of this model, the relationship between the shape of the velocity time–curve, the liquefied layer thickness, and the surface ground slope was investigated, thereby clarifying the mechanisms involved in liquefaction-induced lateral spreading.

Recently, our group used an SPH method to analyze flow processes in liquefied soils (Huang et al., 2011b). A Bingham model combined with the Mohr–Coulomb yield criterion, the concepts of equivalent Newtonian viscosity, and the Verlet neighbor list method were introduced into an SPH framework to develop an algorithm for the analysis of flowing liquefied soils. This model was able to consider the phase conversion of the subsoil from a solid to a kind of viscous liquid during liquefaction, bridging the gap between solid and liquid. In the simulation of a physical model test, the flow processes of liquefied soils were reproduced, and dynamic behavior (e.g., the flow configuration, velocity, and distance) was constrained. Fig. 4 shows the simulated velocity distribution in the liquefied soil. Depending on the dynamic behaviors of the materials involved, designs could be implemented to improve the seismic safety of structures.

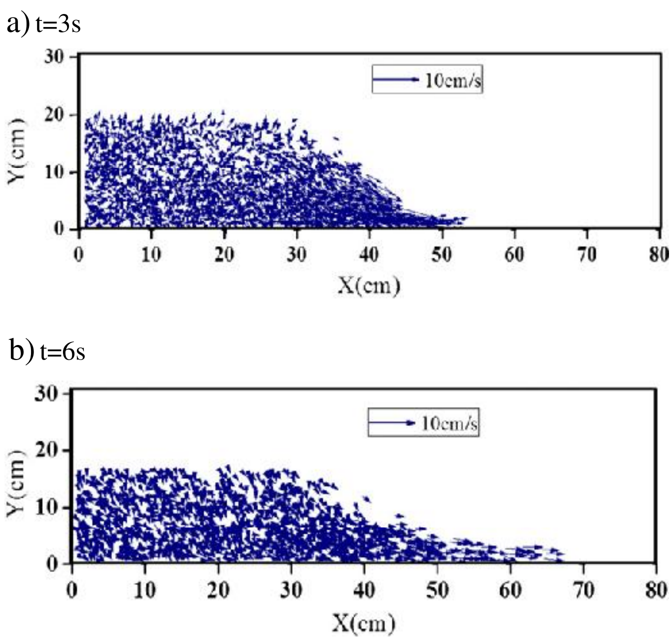

Fig. 4. Velocity distribution within liquefied sand (based on Huang et al., 2011b).

Naili et al. (2005b) used an SPH method to simulate the flow of liquefied soil around a model pile to analyze the drag force applied by liquefied flow. The pile was discretized by a series of particles exerting a Lennard–Jones potential on the surrounding medium. The liquefied soil was again assumed to be a Bingham fluid, and a bilinear model was introduced to consider the recovery of rigidity. The proposed model can reproduce the external configuration of the soil after liquefaction, the distribution of flow velocities, and the strain and stress field of the liquefied soil around the pile. However, both the soil–structure interaction at the interface and the deformation and failure processes of the pile will require more attention in future research.

Even though the liquefied material is a mixture of water and soil, it is assumed to be a Bingham fluid in the above-mentioned models. Therefore, interactions between the solid and liquid parts of the liquefied material could not be taken into consideration. Also, the dissipation of pore water pressure and the recovery of the soil strength could not be addressed. As a result, the modeled amplitudes of flow velocities were smaller than the observed ones, and the modeled flow velocity decreased at a slower rate than the experimental one.

Despite the flaws, the existing SPH models can predict the lateral spread of liquefied soil and calculate the destructive force within the margin of error, which are essential to the design of foundations and surface substructures in seismic zones.

3.4. Seepage

Large-scale deformation and hydraulic collapse of the ground, induced by water flow through the ground, play significant roles in the destabilization of dam foundations during floods, liquefaction, and other catastrophic events. To analyze these phenomena, Japanese researchers introduced SPH method as a way to combine both discrete and continuum techniques for the analysis of ground failure linked with seepage (Maeda and Sakai, 2004). An SPH model that was able to consider the interactions among all three phases (soil, water, and air) was proposed to model progressive seepage failure in soil. In this model, the solid and fluids are considered to be in different layers. To combine layers of different phases, mixture theory was used to calculate the frictional body forces resulting from the velocity differences between adjacent phases. The application of this modeling method led to a numerical simulation of seepage processes around a sheet pile. The evolution of air bubbles during the seepage process was reproduced and the deformation and failure of the ground as induced by the air bubbles was successfully predicted.

Some improvements to the SPH method have been proposed in the course of this research. For example, a new procedure for calculating density sums up different materials from each given phase, thereby making it suitable for problems concerned with the interfaces between different geological materials. Another highlight of this research is accounting for solid–water–air bubble interactions using the SPH method, and explaining the phenomenon of gas generation and air bubble blow-out which are regularly observed to be associated with seepage failures.

3.5. Dynamic erosion

Dynamic erosion is the process by which soil and rock are removed from the earth's surface under the influence of external factors such as water flow, and then transported and deposited in other locations. Excessive erosion may result in problems such as desertification, decrease in agricultural productivity due to land degradation, and ecological collapse due to loss of the nutrient-rich upper soil layers. With its advantages in simulation of flows involving rapid and large displacements, free-surfaces, and moving interfaces, the SPH method becomes an effective technique for numerical modeling of dynamic erosion.

Kristof et al. (2009) presented a visual hydraulic erosion model based on fully 3D water dynamics. It efficiently couples fluid simulation using the SPH method, and a physically-based erosion model adopted from an Eulerian approach. Boundary particles were used to handle the interactions (e.g., friction, sediment erosion, and deposition), and mediate sediment exchange between the moving fluid and underlying terrain. Two numerical examples, including lake water eroding away from the boundary and waterfalls eroding the underlying terrain, were studied to demonstrate the accuracy of the present method. The results showed that the SPH method is efficient for modeling the erosion of dense, large, and sparse fluid. In order to investigate the hydraulic erosion features on levees, dams, and earth embankments in storms and floods, SPH simulations of hydraulic erosion were conducted (Chen et al., 2011). In the simulations, fluid behavior was modeled by the Navier–Stokes equation while the terrain was represented as a segmented height field (SHF). The erosion function was introduced with a critical shear stress to calculate the shear stress applied on soil particles by water flows and to define the minimum shear stress that can results in erosion. This model was verified by comparing the simulation results with the physical test results. The formation of a gulley on the levees when overtopped was reproduced and a better understanding of the erosion process was obtained, providing valuable knowledge for levee engineers. More recently, Manenti et al. (2012) proposed an SPH-based numerical model for the prediction of the coupled water–sediment dynamics induced by the rapid water flow in an artificial reservoir. In their model, both water and soil were assumed as weakly compressible viscous fluids governed by Navier–Stokes equations. Two erosion criteria, based on the Mohr–Coulomb yielding criterion and the Shields theory were introduced for the description of the failure mechanism of sediments. This model was validated by comparing the numerical results with laboratory test data.

3.6. Underground explosions

Blast loading from an accidental explosion, blast excavation, or weapon attacks, can cause large deformation of soil and rock, and seriously affect the safety and stability of buildings and structures nearby. Therefore, the response of soil and subsurface structures subjected to explosion loading has attracted much interest in current protective engineering research. Numerical simulations have become the main tool in such assessments because of the difficulty and expense of conducting large-scale field explosion tests. However, a realistic computation can be difficult because of the extremely large deformation involved and the need for complex modeling of the interactions between the modeled explosive detonations and the soil and buried structures. This fact has encouraged research into applications of SPH methods in this field. Wang et al. (2005) proposed a coupled SPH-FEM approach to simulate the response of soil and buried structures to blast loading. The large-scale soil deformation processes in the vicinity of the explosive charge and the response of the remaining low-deformation regions were reproduced by SPH and FEM solutions, respectively. The SPH particles and FEM elements were joined together on the interface as shown in Fig. 5. On that basis, the 2D model was developed into a 3D model (Lu et al., 2005), and comparisons between the different response features obtained by 2D and 3D models were undertaken. The propagation of a blast wave around edges and corners is much more complicated in 3D models, thus largely influencing the loading conditions and the response of the structure. More recently, the modeling method has been applied to analyze liquefaction mechanisms induced by blast loading and to investigate the effect of soil liquefaction on surface structures (Wang et al., 2011). A three-phase soil model for shock loading was proposed, and the interface interactions among the explosive detonation, soil medium, and geological structures were modeled using a completely joined surface. The time history plots of pore water stress were obtained, and the extent of liquefaction areas was predicted (which again coincided with empirical predictions). A conclusion was drawn stating that the surface structures would remain stable with limited permanent settlement if the liquefaction zone does not extend into the foundation region.

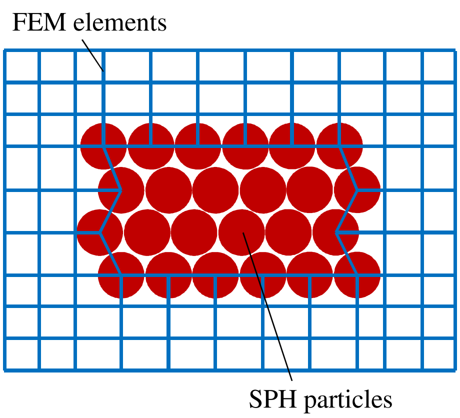

Fig. 5. Coupled mesh of SPH particles and FEM elements (based on Wang et al., 2005).

In the coupled SPH-FEM model of Xu and Liu (2008), the master–slave algorithm proposed by De Vuyst et al. (2005) was introduced to account for the contact interaction of FE and SPH particles. Blast analysis was conducted using the Jones–Wilkins–Lee (JWL) equation of state for high explosives. The accuracy of this coupled approach, and its advantage over the simulation of larger deformations, were validated by free field blast analysis. In the application of this modeling method to a subsurface blast, all the processes involved the formation of explosive craters. This required the propagation of pressure and the response of structures to be reproduced.

It can easily be concluded that SPH applications applied to underground explosions have achieved significant outcomes. These include: 1) the coupled SPH-FEM approach to reproduce the dynamic response of soils and structures, 2) the three-phase soil modeling technique, which is capable of modeling the internal distribution of stresses among the soil components and describing changes in the pore water pressure, 3) blast analysis based on an equation of state to simulate interactions between the shock wave and the structure, 4) modeling of mechanisms of soil liquefaction caused by blast loading and the effect of the liquefied soil on structures, and 5) stability criteria for surface structures subjected to underground explosions. These outcomes have important roles in the design of underground explosion protection systems and in minimizing the damage caused by blast vibrations.

Blasts are extremely complex processes in which the explosive charge experiences a violent chemical reaction and conversion into gaseous products at very high temperatures and pressures. In the simulations mentioned above, blast analyses were conducted based on an equation of state containing some assumptions; the validation is also based on empirical equations. Therefore, additional research is needed and the reliability of the proposed SPH model should be further verified, especially in the burst mode.

3.7. Rock breakage

Rock caving is a large deformation problem that is often combined with the formation of crucial fractures and fragmentation. Karekal et al. (2011) established an SPH solution for elastic solid deformation, combined with a modified damage mode based on the local stress history and flaw distribution, to simulate rock caving processes. This method has had success in: a) identifying rock deformation, fracture formation, and fragmentation processes that lead to progressive roof collapse, b) characterizing the level of fracturing at a specific location, and c) capturing the elastic–brittle and elastoplastic material behavior of a rock mass. The coupled SPH-FEM simulations of rock caving conducted as part of this research have been proven to be computationally efficient.

Das and Cleary (2010) used an SPH approach with a continuum damage model to simulate rock breakage under impact. A numerical experiment to test unconfined compressive strength (UCS) was conducted as a validation of the SPH-based damage model for predicting rock fracture; the modeled and experimental results were in good agreement. Subsequently, this model was used to simulate the brittle fracture of rock specimens of different shapes during impact. This led to the conclusion that rock shape has considerable influence on the fracture process, fragment sizes, energy dissipation, and post-fracture motion of the fragments.

Ma et al. (2011) developed an SPH code to simulate uniaxial and biaxial compression tests used for studying the mechanical characteristics and failure processes of heterogeneous rock-like materials. The evolution of the failure was well captured; cracks and fragments with large deformations were easily reproduced. The influences of material heterogeneity and confining load conditions were investigated.

The application of SPH methods to rock breakage embodies its superiority with regard to the simulation of extremely non-linear physical processes such as fracture and fragmentation. The prediction of rock-breakage characteristics assists in our understanding of the fundamentals of rock failure and improves structural designs. However, unlike landslides, explosions, and other macro-scale disasters, the formation and development of cracks in geological materials can be significantly affected by micro-structural properties. A full understanding of the mechanics at such fine scales is limited by the resolution of this macro-scale method. Multi-scale methods constitute a class of simulation techniques that bridge the gap between the nano-, micro-, meso- and macro-scales of physical processes. Such methods have become efficient and robust procedures for investigating complex physical phenomena. Therefore, developing a coupled SPH model for multi-scale simulations will be a priority for future geotechnical research.

4. Conclusions and prospects

Geo-disasters could easily lead to casualties and large-scale economic losses. Reproducing their evolution process is of importance for prediction and prevention of such disasters. This paper demonstrates that the SPH method has been successfully applied to many geo-disasters, including dam breaks and coastal engineering, slope failure and landslides, soil liquefaction, seepage failure, dynamic erosion, underground explosions, and rock breakage. Perhaps the most successful and well-established applications involve dam breaks. Other applications (such as soil liquefaction and underground explosions) need to be considered with caution until additional research has been undertaken. From the achievements of the studies reviewed in this paper, the SPH method has significant advantages over traditional grid-based numerical modeling methods when dealing with geo-disasters concerning extremely large deformation, free surfaces, moving interfaces, and deformable boundaries. This suggests that SPH is a promising method for studying geo-disasters.

In previous research, the SPH method was applied to various types of computational fluid dynamics problems without considering the physical strength of the materials involved. For geo-disasters applications, the constitutive model and the equation of state for materials with physical strength have been imported into the SPH framework. Although this method usually shows stress instabilities when applied to solid mechanics, some effective solutions have been proposed. Therefore, for geo-disaster applications, the SPH method has evolved from a hydrodynamics technique to a mechanics technique.

Geo-disasters involve many intricate problems concerning complex material interfaces between different geological materials. In the research reviewed here, effective solutions have been proposed to address the interaction between different geological materials. These include: the interaction of ocean waves with man-made structures in coastal engineering, the coupling of multiphase flow in dam breaks, the interface between water and underwater landslide materials, solid–water–air bubble interactions in seepage failure, and the interactions between an explosive charge and the surrounding overburden and geological structures in an underground explosion.

The technique of producing hybrid mixes of different numerical methods is currently popular in computational mechanics research. In geo-disasters, coupled SPH-FEM models have been widely applied, thus improving computing efficiency and extending the application of the SPH method. Some Eulerian techniques have been incorporated into SPH models, such as the VOF method and the LS approach. In addition, some advanced SPH methods (e.g., incompressible SPH, corrected SPH, discontinuous SPH) have enhanced the stability and accuracy of the conventional SPH method.

Despite its good performance in many situations, SPH still suffers from a number of shortcomings, notably: 1) imprecision caused by the lack of particle coverage near boundaries, 2) tensile instability when handling problems with material strength, and 3) zero-energy modes when field variables and their derivatives are calculated at the same locations. In addition, most simulations in existing research are based on simplified 2D axisymmetric models that may cause inaccuracy or even errors in some cases. A 3D analysis approach is required for a more realistic simulation. However, SPH is a computationally intensive numerical method. Because 3D simulations require a large number of particles throughout the SPH region, they have extensive computer-memory needs and require longer computational run times. Simplifying the computational algorithm is a preferred way to improve calculation efficiency. On the other hand, parallel computing options could provide the needed computer time savings. They involve novel technical approaches that make use of simultaneous multiple computing resources. A few efficient parallel algorithms have already been embedded in SPH codes, and the calculation efficiency was improved to a great extent (Wroblewski and Boryczko, 2008; Hubber et al., 2011; Marrone et al., 2012). Further development of parallel algorithm use in SPH can expand its application to large computational geotechnical problems.

Geological materials are complex particulate media with a wide range of mechanical properties at different scales. For a full understanding of the processes, many mechanical phenomena, such as the formulation and development of cracks, should be interpreted at a wide range of scales. Future research into the SPH method will need to couple such processes with molecular-level simulations (such as those involved with molecular dynamics), to further expand the range of multi-scale analyses in geotechnical engineering problems.

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